In control theory and stability theory, root locus analysis is a graphical method for examining how the roots of a system change with variation of a certain system parameter, commonly a gain within a feedback system. This is a technique used as a stability criterion in the field of classical control theory developed by Walter R. Evans which can determine stability of the system. The root locus plots the poles of the closed loop transfer function in the complex s-plane as a function of a gain parameter (see pole–zero plot).

USES

Effect of pole location on a second order system's natural frequency and damping ratio.

In addition to determining the stability of the system, the root locus can be used to design the damping ratio (ζ) and natural frequency (ωn) of a feedback system. Lines of constant damping ratio can be drawn radially from the origin and lines of constant natural frequency can be drawn as arcs whose center points coincide with the origin. By selecting a point along the root locus that coincides with a desired damping ratio and natural frequency, a gain K can be calculated and implemented in the controller. More elaborate techniques of controller design using the root locus are available in most control textbooks: for instance, lag, lead, PI, PD and PID controllers can be designed approximately with this technique.

The definition of the damping ratio and natural frequency presumes that the overall feedback system is well approximated by a second order system; i.e. the system has a dominant pair of poles. This is often not the case, so it is good practice to simulate the final design to check if the project goals are satisfied.

DEFINITION

The root locus of a feedback system is the graphical representation in the complex s-plane of the possible locations of its closed-loop poles for varying values of a certain system parameter. The points that are part of the root locus satisfy the angle condition. The value of the parameter for a certain point of the root locus can be obtained using the magnitude condition.

Suppose there is a feedback system with input signal X(s) and output signal Y(s). The forward path transfer function is G(s) the feedback path transfer function is H(s). The closed loop transfer function G(s) is given by;

T(s) = Y(s)/X(s) = G(s)/1+G(s)H(s).

Thus, the closed-loop poles of the closed-loop transfer function are the roots of the characteristic equation 1+G(s)H(s) = 0. The roots of the equation can be found wherever G(s)H(s) = -1.

In systems without pure delay, the product is a rational polynomial function and may be expressed as

where –Z1 are the M zeros, -Pi are the N poles, and K is a scalar gain. Typically, a root locus diagram will indicate the transfer function's pole locations for varying values of the parameter K. A root locus plot will be all those points in the s-plane where G(s)H(s)=-1 for any value of K.

The factoring of K and the use of simple monomials means the evaluation of the rational polynomial can be done with vector techniques that add or subtract angles and multiply or divide magnitudes. The vector formulation arises from the fact that each monomial term (s-a) in the factored G(s)H(s) represents the vector from a to s in the s-plane. The polynomial can be evaluated by considering the magnitudes and angles of each of these vectors.

According to vector mathematics, the angle of the result of the rational polynomial is the sum of all the angles in the numerator minus the sum of all the angles in the denominator. So to test whether a point in the s-plane is on the root locus, only the angles to all the open loop poles and zeros need to be considered. This is known as “ANGLE CONDITION”.

Similarly, the magnitude of the result of the rational polynomial is the product of all magnitudes in the numerator divided by the product of all the magnitudes in the denominator. It turns out that the calculation of magnitude is not needed to determine if a point in the s-plane is part of the root locus because K varies and can take arbitrary real values. For each point of the root locus, a value of K can be calculated. This is known as the
“MAGNITUDE CONDITION”.
The root locus only gives the location of the closed loop poles as the gain K is varied. The value of K does not affect the location of the zeros. The open loop zeros are the same as the closed loop zeros.

Angle condition: a point s of the complex s-plane satisfies this condition if

$$\Delta(G(s)H(s)) = \pi$$

which is the same as saying that the sum of angles from the open loop zeros to the point s minus the angles from the open loop poles to the point s has to be equal to π or 180 degrees.

Magnitude condition: the value of K satisfies the magnitude condition for a given s point of the locus if

$$|G(s)H(s)| = 1.$$

SKETCHING ROOT LOCUS

Using a few basic rules, the root locus method can plot the overall shape of the path (locus) traversed by the roots as the value of K varies. The plot of the root locus then gives an idea of the stability and dynamics of this feedback system for different values of K. The rules are the following:

Mark open-loop poles and zeros

Mark real axis portion to the left of an odd number of poles and zeros

Find asymptotes

Let P be the number of poles and Z be the number of zeros:

P-Z= number of asymptotes

The asymptotes intersect the real axis at α (which is called the centroid) and depart at angle Ø given by:

$$Øl = 180^0 + (l-1)360^0 / P-Z$$

$$\alpha = \sum p - \sum z / P-Z$$

where ∑p is the sum of all the locations of the poles, and ∑z is the sum of all the locations of the explicit zeros.

Phase condition on test point to find angle of departure

Compute breakaway/break-in points

The breakaway points are located at the roots of the following equation:

$$dG(s)H(s)/ds=0.$$

Once you solve for z, the real roots give you the breakaway/reentry points. Complex roots correspond to a lack of breakaway.

The root locus method can also be used for the analysis of sampled data systems by computing the root locus in the z-plane, the discrete counterpart of the s-plane. The equation $z = \exp ST$ maps continuous s-plane poles (not zeros) into the z-domain, where T is the sampling period. The stable, left half s-plane maps into the interior of the unit circle of the z-plane, with the s-plane origin equating to $|z| = 1$ (because $e0 = 1$). A diagonal line of constant damping in the s-plane maps around a spiral from (1,0) in the z plane as it curves in toward the origin. Note also that the Nyquist aliasing criteria is expressed graphically in the z-plane by the x-axis, where $\omega nT = \pi$. The line of constant damping just described spirals in indefinitely but in sampled data systems, frequency content is aliased down to lower frequencies by integral multiples of the Nyquist frequency. That is, the sampled response appears as a lower frequency and better damped as well since the root in the z-plane maps equally well to the first loop of a different, better damped spiral curve of constant damping. Many other interesting and relevant mapping properties can be described, not least that z-plane controllers, having the property that they may be directly implemented from the z-plane transfer function (zero/pole ratio of polynomials), can be imagined graphically on a z-plane plot of the open loop transfer function, and immediately analyzed utilizing root locus.

Since root locus is a graphical angle technique, root locus rules work the same in the z and s planes.

The idea of a root locus can be applied to many systems where a single parameter K is varied. For example, it is useful to sweep any system parameter for which the exact value is uncertain in order to determine its behavior.

^ Kuo 1967, p. 331.

^ Kuo 1967, p. 332.

^ Evans, Walter R. (1965), Spirule Instructions, Whittier, CA: The Spirule Company

^ Evans, W. R. (January 1948), "Graphical Analysis of Control Systems", Trans. AIEE, 67 (1): 547–551, doi:10.1109/T-AIEE.1948.5059708, ISSN 0096-3860

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