EXISTENCE OF MINIMIZERS IN FINITE DIMENSIONAL SPACES

BY

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Certification

This is to certify that this project on, **Existence of Minimizers in Finite Dimensional Spaces** was carried out by **Okoroafor, Promise C.** with registration number **2013/190716** of the Department of Mathematics, Faculty of Physical Sciences, University of Nigeria, Nsukka.

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Dedication

I dedicate this research work to **God**, for His unimaginable support and tender compassion in my life throughout my stay as an undergraduate.

Acknowledgment

I am very grateful to God Almighty for His infinite mercies and the grace He granted me to participate in this grueling yet fruitful academic exercise.

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Abstract

This research paper seeks to provide a general way of doing optimization, which is pertinent when we deal with minimization in finite dimensional spaces.

In this work basic definition were provided, concise proofs and relevant examples were given. Some optimization theorems were stated alongside their proofs in conclusion.

Contents

Chapter 1

Introduction

Mathematics,a subject of discussion amongst scholars,theorists and scientists alike on the basis of its applicability,is one of the critical issues.This difficulty is owing to the fact that mathematics is a subject of theorems,propositions,lemmas,corrolaries explaining a basic concept,which may not be well understood to the non-mathematicians and to the learned with non-curiosity. There is a burning debate among non-mathematicians regarding the usefulness of some aspects of mathematics. It is however an undisputable fact that mathematics is the foundation of most scientific endeavor,be it in the art of signs and figures.

However, optimization theory is one of the areas where some mathematical concept find their application, be it minimization of cost or maximization of gain. In developing theorems that are used in optimization one encounters some constraint either on the objective function or on the space.

Finite dimensional spaces are encountered in numerous ways. The aim of this work is to discuss the general way of minimizing certain functionals in this space forthwith.

To do this, we are going to consider conditions that can enable minimization to be carried out on these spaces.

1.1 Scope of study

In this work,we are interested in the existence of a minimizer for a functional on finite dimensional spaces,where we considered this on a normed linear space,hence the theorems proved in this work guarantees such existence. Since some real life problems normally results in finite dimensional spaces,we should consider minimizing negative effects of these problems.Hence,we will be defining minimization of the space with respect to the space being closed and bounded,that is,compact.

1.2 Limitations of the study

This research work did not exhaust all literatures and as well as few finite dimensional spaces such as $\Re^n$ were considered .Typing this work using Texmaker,a version of Latex used for typing mathematical projects,articles e.t.c.,was quite difficult for me as I found it hard to understand some of the commands and programs necessary for typesetting as very few people knew the rudiments of the software.However,with much curiosity and seriousness,I overcame.

1.3 Aims and Objectives

This work aims at enabling readers to effectively minimize certain functionals on finite dimensional spaces by understanding and effectively explaining and proving the concepts of minimization and its associated theorems in finite dimensional optimization.

1.4 Motivation of study

In certain optimization problems,the unknown optimal solution might not be a number or a vector but rather a continuous function,for example the size and shape of small parts of a rigid body,before considering the body as a whole.Such is finite dimensional optimization as continuity of each of the small parts may not be determined by infinite dimensional optimization,as it may be difficult to consider in that part.The subdivision of a whole domain into simpler parts has several advantages:

1. Accurate representation of complex geometry
2. Inclusion of dissimilar material properties
3. Easy representation of the total solution
4. Capture of local effects

1.5 Definition of Background Terms and Concepts

We define basic terms one must know or have known before continuing with other chapters in this work,as they are the basics in mathematical analysis.

1.5.1 Limit of a function

Let c be a point in some interval I of the real line $\Re$.Let f be a function which is defined at every point of I except possibly at c.The limit of the function f as x approaches c is L,written

$$lim_{x \longrightarrow c} f(x) = L(or\ f(x) \longrightarrow L\ as\ x \longrightarrow c)$$

if for any positive number ε(no matter how small) there is some $\delta > 0$ such that $|f(x) - L| < \varepsilon \ for\ all\ 0 < |x - c| < \delta$.

Examples are $lim_{x \longrightarrow 6}(3x - 4) = 14$,$lim_{x \longrightarrow 0}|x| = 0$,$lim_{x \longrightarrow 2}\frac{x+1}{3x+4} = \frac{3}{10}$ e.t.c

1.5.2 Continuity of a real valued-function

Let $f : D(f) \subseteq \Re \longrightarrow \Re$ be a map with $x_0 \in D(f) \subseteq \Re$; f is said to be continuous at $x_0 \in D(f)$ if given any $\varepsilon > 0, \exists\ a\ \delta = \delta(\varepsilon) > 0$ such that $\forall x \in D(f)$

$$|f(x) - f(x_0)| < \varepsilon \text{ whenever } |x - x_0| < \delta.$$

Examples of continuous functions are all polynomial functions,the function $f(x) = sin(\frac{1}{x})$ on $(0, \frac{1}{\Pi})$,the function $f : [0, \infty) \longrightarrow \Re$ defined by $f(x) = \sqrt{x}$ is continuous at 0 e.t.c

We note the following theorem concerning continuity which will be used in proceeding chapters.

Theorem 1.53

Let f be a real-valued function defined on some interval around $x_0 \in D(f)$.Then f is continuous at x_0 if and only if

$$lim_{x \longrightarrow x_0} f(x) = f(x_0).$$

1.5.3 Uniformly continuous functions

A real-valued function f is said to be uniformly continuous on a set S if

(i) $S \subset dom(f); and,$

(ii) $\forall \varepsilon > 0, \exists \delta = \delta(\varepsilon) > 0$ such that,$\forall x_1, x_2 \in S$,

$$|x_1 - x_2| < \delta \Rightarrow |f(x_1) - f(x_2)| < \varepsilon.$$

Examples include the function $g : (0, 1) \longrightarrow \Re$ by $g(x) = x^2$ is uniformly continuous,$f(x) = cosx$ and $sinx$ on $-\infty < x < \infty$ e.t.c

1.5.4 Sequences and Subsequences

A sequence $\{a_n\}$,$n \geqslant 1$ of real numbers is a function a:

$D(a) \subset \mathbb{N} \longrightarrow \Re$ of an infinite subset $D(a)$ of the natural numbers $\mathbb{N}$ into the real numbers $\Re$ defined by

$$a_n = a_n \in \Re \text{ for all } n \in \natural N.$$

For the definition of a subsequence,

let $\{a_n\}$ be a sequence in X and let $\{n_k\}$ be any sequence of natural numbers such that $n_1 < n_2 < n_3 < ...$The sequence $\{a_{n_k}\}$,$k \geqslant 1$ is called a subsequence of $\{a_n\}$.

1.5.5 Convergent sequence

A sequence $\{a_n\}$ of real number is said to converge to a real number a^* if and only if $\forall \varepsilon > 0$ *given*, $\exists$ *a natural number n* $\in$ $\mathbb{N}$ *such that*

$$|a_n - a^*| < \varepsilon \text{ for all } n > n(\varepsilon).$$

1.5.6 Closed set

A set $F \subseteq \Re$ is said to be closed if it contains the limit of all its convergent sequences.i.e.,a set is closed if it contains all its limit points.

Example 1.58

The interval $[a, b]$ is closed in $\Re$.

Solution: Let $\{x_n\}$ be an arbitrary sequence in $[a, b]$ such that $x_n \longrightarrow x$ as $n \longrightarrow \infty$.It suffices to prove $x \in [a, b]$ implies,$a \leqslant x_n \leqslant b$,$\forall$ $n \in \mathbb{N}$.Since $lim x_n$ exists,we have

$$limit\ a \leqslant limit\ x_n \leqslant b, \text{as } n \longrightarrow \infty.$$

This implies $a \leqslant x \leqslant b$ and so $x \in [a, b]$.Hence $[a, b]$ is closed.
Observe that $(0, 1]$ is not closed in $\Re$. For the sequence $\frac{1}{n}$ is in $(0, 1]$ and $\frac{1}{n} \longrightarrow 0$ but $0 \not\subseteq (0, 1]$. So,$(0, 1]$ does not contain all its limit points and so is not closed.

1.5.7 Bounded sets

A subset S of real numbers is said to be bounded from below(or simply bounded below) if there exists $\alpha \in \Re$ such that

$$\alpha \leqslant x \text{ for all } x \in S.$$

The subset S is said to be bounded from above (or simply bounded above) if there exists $\beta \in \Re$ such that

$$x \leqslant \beta \text{ for all } x \in S.$$

The subset S is said to be bounded if it is bounded both from above and from below.In this case,it is customary to say that there exists a constant $M > 0$ such that $|x| \leqslant M$ for all $x \in S$.
We recall that $|x| \leqslant M$,for all $x \in S$ if and only if $-M \leqslant x \leqslant M$,for all $x \in S$.

Example 1.57

(i) Any finite subset of $\Re$ is bounded.

For example,$S = \{-2, -1, 0, 4, 5\}$ is bounded.

Here,$-2 \leqslant x \leqslant 5$ for all $x \in S$ and of course,we can take $M = 5$ and write $|x| \leqslant 5$ for all $x \in S$.

(ii) The set S=$\{\frac{n}{2n+1}, n = 1, 2, 3, ...\}$ is bounded from above(by $\frac{1}{2}$) and bounded from below(by $\frac{1}{3}$).

1.5.8 Infimum of a subset of $\Re$

Let S be a subset of real numbers which is bounded from below.The greatest lower bound for S or the infimum of S(denoted by $inf\ S$)α_0(say) is a real number satisfying the following two conditions:

$$\alpha_0 \leqslant s \forall\ s \in S;$$

$$if \alpha \leqslant s \forall\ s \in S, \qquad then\ \alpha \leqslant \alpha_0.$$

We have an equivalent definition given as follows:

$$\alpha_0 = inf S$$

if and only if the following two conditions are satisfied:

$$\alpha_0 \leqslant s \forall s \in S.$$

$$\forall \varepsilon > 0, \text{ there exists } s_\varepsilon \in S \text{ such that}$$

$$\alpha_0 \leqslant s_\varepsilon < \alpha_0 + \varepsilon.$$

1.5.9 Linear(vector) space

Let X be a non-empty set and K is a field of scalars.Suppose addition (+) and multiplication (*) are defined on X such that X such that X is closed under addition and scalar multplication and such that $\forall x, y \in X$,$\forall \alpha, \beta \in K$,the following are satisfied:

1. $x + y \in X$(Closure).
2. $(x + y) + z = x + (y + z)$(Associativity property).
3. $\exists\ 0 \in X$ such that $x + 0 = 0 + x = x$(Additive identity).
4. $\forall x \in X$,$\exists - x \in X$ such that $x + (-x) = (-x) + x = 0$(Additive inverse).
5. $x + y = y + x$(Commutativity).
6. $\alpha(x + y) = \alpha x + \alpha y$(Distribution of scalars over vectors).
7. $\alpha(\beta x) = (\alpha\beta)x$.
8. $(\alpha + \beta)x = \alpha x + \beta x$(Distribution of vectors over scalars).
9. $1.x = x.1\ \forall\ 1 \in K, x \in X$.

X is called a linear space.If $K = \Re$,then X is called a real linear space.

1.5.10 Basis and Dimension

We say that a set of vectors S spans the vector space V,if every vector in V can be written as a linear combination of some vectors in S.This means that if $S = (V_1, V_2, ..., V_k)$ say,then for every $v \in V$,there exists scalars $c_1, c_2, ..., c_k \in \Re$ such that $v = c_1V_1 + c_2V_2 + ... + c_nV_n$.

By a basis for a vector space V,we mean a set of vectors,say $S = (V_1, V_2, ..., V_k)$, satisfying the following two conditions:

1. The set spans V.

2. The set is linearly independent.

Any two bases for a vector space V contain the same number of vectors.This number is called the dimension of the vector space V.Thus,by dimension of a vector space V,we mean the number of vectors in a basis.

1.5.11 Finite Dimensional vector space

We say that a vector space V is finite dimensional if the number of vectors in a basis is finite.That is, $dim(V) \leqslant n.$

For example,the dimension of $\Re^2$ is 2, dimension of $\Re^3$ is 3 and that of $\Re^n$ is n.

Chapter 2

Literature Review

Optimization is a dynamic and widely applied mathematical discipline. Properties of minimizer and maximizer of functions,which are the basic concepts of optimization depends greatly on rich and thriving concepts from mathematical analysis, including techniques from calculus and its generalization, topological notions and more geometric ideas. The theory underlying current optimization techniques,especially as to the concept of minimization grow ever more sophisticated over time.

2.1 Historical Development of Optimization.

The existence and usage of optimization methods can be traced to the days of Euclid,Heron,Newton,Lagrange and Cauchy.The development of differential calculus methods for optimization was possible because of the contributions of Newton and Liebnitz to calculus. The foundation of calculus of variations, which also deals with the minimization of functions, was laid by Bernoulli, Euler, Lagrange, and Weierstrass. Cauchy made the first application of the steepest descent method to solve unconstrained optimization problems. By the middle of the twentieth century, the high speed digital computer made implementation of the complex optimization procedures possible and stimulated further research on newer methods. Spectacular advances followed, producing a massive literature on optimization techniques. This advancement also resulted in the emergence of several well defined new areas in optimization theory

Some of the major developments in the history of optimization are outlined here with few milestones.
Euclid(300bc) considers the minimal distance between a point and a line, and proves that a square has the greatest area among the rectangles with given total length of edges.
Heron(100bc) proves in cataprica that light travels between two points through the path with shortest length when reflecting a mirror.
J. Kepler(1615) figures out the optimal dimensions of wine barrel. He also formulated an early version of the secretary problem (a classical application of dynamic programming).
P. de Fermat(1657) shows that light travels between two points in minimal time.

Isaac Newton (1660s) and G.W. Von Leibintz (1670s) created mathematical analysis that forms the basis of calculus of variation. Some separate finite optimization problems were also considered.

Newton(1687) studied the body of minimal resistance.

Johann and Jacob Bernolli(1696) studied Brachristochrone's problem, which gave birth to calculus of variation.

L. Euler's(1740) publication began the research on general theory of calculus of variation.

P.L.M. de Manupertuis(1746) formulated the principle of least action to explain physical phenomena.

J.L. Lagrange(1760) formulated the plateau's problem of minimal surface.

G. Monge(1784) investigated a combinational optimization problem known as the transportation problem.

A.M. Legendre(1806) presented the least square method.

J.B.J. Fourier(1826) formulated LP-problem for solving problem arising in mechanics and probability.

A.L. Cauchy(1847) presented the gradient method.

T.W. Gibbs(1857) showed that chemical equation is an energy minimum.

G. Dantzgig(1947) who worked for US air force, presented the simplest method for solving LP problems.

H.W. Kuhn and A.W Tucker(1951) reinvented optimality conditions for non-linear problems.

L.S.Pontryagins(1956) research group presented maximum principle.

R. Bellman(1957) presented the optimality principle.

Zoutenduk(1960s) presented the methods of feasible directions to generalize the simplest method for non-linear programs.

Heunstic(1980s) algorithm for global optimization and large scale problem began to give popularity.

Irwin Schochetman E. and Robert Smith L.(2001) developed a finite algorithm used for solving infinite dimensional optimization problems

Mordukhovich Boris S.(2007) on variational analysis in non-smooth optimization and discrete optimal control.

Bayraktar Erhan and Egami Masahiko(2010) discussed the one-dimensional optimal switching problem.

Mordukhovich B.S.,Nghia T.T.A. ,and Rockafellar R.T(2015) on full stability in finite dimensional optimization.

2.2 Historical Development of Optimization on Finite Dimensional spaces.

The concept of Finite Dimensional Spaces was introduced by George Hamel (1901) and called "finite dimensional" by H.Haln (1927). Hahn recognized the importance of finite dimension in his study of linear equations in normed spaces which was motivated by integral equations and also contains the Hahn-Banach theorem as well as the earliest investigation of dual spaces. The emergence of finite dimensional spaces sprout forth theorems that guarantees both the existence of a minimizer or maximizer of an objective functional on this space which induces a more general way of optimization in this discourse.

However,researches on the existence of minimizers on this space was supported by Weierstrass and Berkowitz,L.D (2002),where he proved the existence of minimizers by invoking some classical theorems of real analysis,which he called "the fundamental existence theorem for finite dimensional optimization".

Chapter 3

Fundamental Concepts and Definitions of Minimizers in Finite Dimensional Spaces

In this chapter,we are going to give basic concepts and definitions showing that on finite dimensional spaces,the existence of minimizers are guaranteed by certain conditions.We note mainly that on finite dimensional spaces,closed and bounded subsets are compact.Nevertheless,we discuss this space with respect to compactness.To this end,we start with the following theorem which is an implying concept for other definitions in this discourse.

Theorem 3.0.0 Let $F : [a, b] \longrightarrow \Re$ be continuous.Then,

(i) F is bounded on $[a, b]$.

(ii) There exists a point c $\in [a, b]$ at which F attains a maximum value,i.e.,

there exists c $\in [a, b]$ such that

$$\text{F(c)}=max_{x\in[a,b]}F(x).$$

(iii) There exists a point $c^* \in [a, b]$ at which F attains a minimum value,i.e.,there exists $c^* \in [a, b]$ such that

$$F(c^*)=min_{x\in[a,b]}F(x).$$

(iv) $F([a, b]) = [F(c^*)\ , F(c)]$.

(v) F is uniformly continuous.

However,having these basic concepts at the back of our minds,we can now define the following concepts concerning the existence of minimizers in finite dimensional spaces.

For possible consideration of minimizers on this space,we look at the special case where the space is normed linear.So we now define with possible examples a normed linear space.

3.1 Normed Linear Space

3.1.1 Definition

Let X be a linear(vector) space over the scalar field K.A norm,$\|.\|$,on X is a non-negative real valued function written as $\|.\| : X \longrightarrow [0, \infty)$ satisfying the following conditions:

(i)

$$a)\|x\| \geqslant 0 \forall x \in X.$$

$$b)\|x\| = 0 \; iff \; x = 0.$$

(ii) $\|\lambda x\| = |\lambda|\|x\| \forall x \in X, \lambda \in K.$

$(iii)\|x + y\| \leqslant \|x\| + \|y\| \; \forall x, y \in X.$

So the pair $(X, \|.\|)$ is a normed linear space.

Example 3.12

Let $X = \Re$.Define $\|.\| : \Re \longrightarrow [0, \infty)$ by $\|x\| = |x| \; \forall x \in X$.Then $\|.\|$ is a norm on $\Re$.

Proof: (i) a) $\|x\| = |x| \geqslant 0$ by definition of absolute value $|.|$ $\forall x \in \Re$.

b)

$$If \; \|x\| = |x| = 0, then \; x = 0$$

$$On \; the \; other \; hand if \; x = 0, then$$

$$\|x\| = |x| = |0| = 0.$$

(ii) $\|\lambda x\| = |\lambda x| = |\lambda||x| = |\lambda|\|x\| \; \forall x \in X, \lambda \in K.$

$(iii)\|x + y\| = |x + y| \leqslant |x| + |y| = \|x\| + \|y\| \; \forall x, y \in X$

Therefore $\|x\| = |x|$ is a norm on $\Re$ and $(\Re, \|.\|)$ is a normed linear space.This particular $\|.\| = |.|$ is called the usual norm on $\Re$.

Example 3.13

Let $X = \Re^2$.Define $\|.\|_1 : \Re^2 \longrightarrow [0, \infty)$ by $\|\overline{x}\|_1 = \sum_{i=1}^{2} |x_i| \; \forall \overline{x} \in \Re^2$.Then,$\|.\|_1$ is a norm on $\Re^2$

Proof: (i) a) $\|\overline{x}\|_1 = \sum_{i=1}^{2} |x_i| = |x_1| + |x_2| \geqslant 0$ by definition of absolute value and sum $\forall \overline{x} \in \Re^2$.

b)

$$Assume\ \|\overline{x}\| = 0\ and\ show\ that\ \overline{x} = 0.$$

$$But\ \|\overline{x}\| = 0 \Rightarrow |x_1| + |x_2| = 0$$

$$\Rightarrow |x_1| = 0\ and\ |x_2| = 0$$

$$\Rightarrow x_1 = 0\ and\ x_2 = 0$$

$$\overline{x} = (x_1, x_2) = (0, 0) = 0$$

Next,we assume $\overline{x} = 0$ and show that $\|\overline{x}\|_1 = 0$.

But $\overline{x} = 0 \Rightarrow (x_1, x_2) = (0, 0) \Rightarrow x_1 = 0\ and\ x_2 = 0$.

Therefore,$\|\overline{x}\|_1 = |x_1| + |x_2| = |0| + |0| = 0$.

(ii)

$$\begin{aligned}\|\lambda\overline{x}\|_1 &= \|\lambda(x_1, x_2)\|_1 \\ &= \|(\lambda x_1, \lambda x_2)\|_1 \\ &= |\lambda x_1| + |\lambda x_2| \\ &= |\lambda||x_1| + |\lambda||x_2| \\ &= |\lambda|[|x_1| + |x_2|] \\ &= |\lambda|\|\overline{x}\|_1.\end{aligned}$$

(iii)

$$\begin{aligned}\|\overline{x} + \overline{y}\|_1 &= \|(x_1 + y_1, x_2 + y_2)\|_1 \\ &= |x_1 + y_1| + |x_2 + y_2| \\ &\leqslant |x_1| + |y_1| + |x_2| + |y_2| \\ &= |x_1| + |x_2| + |y_1| + |y_2| \\ &= \|\overline{x}\|_1 + \|\overline{y}\|_1\end{aligned}$$

Therefore,$\|.\|_1$ is a norm on $\Re^2$ and $(\Re^2, \|.\|_1)$ is a normed linear space.

With this basic definition and examples given above,we can define norms on finite dimensional spaces,especially the space $\Re^n$.

As stated above,we will be defining minimizers on this space with respect to compactness.So we therefore look at the basic definition of a compact set and consequently important concepts in this set.

3.2 Compact set and its continuous map

Definition 3.21

A set $X \subseteq \Re$ is said to be compact if and only if it is closed and bounded.

Example 3.22

The set $[a, b]$ is compact and the set (a, b) is not.The sets $(-\infty, a]$,$[a, \infty)$ are not.Any finite set is compact.Let $\{x_n\}$ be a convergent sequence and let $x_n \longrightarrow x^*$.Then $X = \{x_n\} \cup x^*$ is compact.

Remark 3.23

If $X \subseteq \Re$ is compact,then the following statements are immediate:

(i) Any sequence in X has a convergent subsequence.

(ii) Any convergent sequence in X has its limit in X.

With this in mind,we now state without proof the following extremely important theorem which considers the behaviour of continuous maps defined on compact subsets of $\Re$.

Theorem 3.2.4 Let $X \subseteq \Re$ be compact and $f : X \longrightarrow \Re$ be continuous.Then

(i) f is bounded.

(ii) f attains its maximum value at some point of X.

(iii) f attains its minimum value at some point of X.

(iv) $f[K]$ is a compact set in $\Re$, i.e., (the continuous image of a compact set is compact)

(v) f is uniformly continuous on X.

With this,we now state the following classical theorem which will be seen subsequently.However,we state without proof.

3.3 The Bolzano-Weierstrass Theorem

Every bounded sequence in $\Re$ has a convergent subsequence.

3.4 Theorem(Uniqueness of limit)

Having seen that a bounded sequence has a convergent subsequence,the question before us now is: can the sequence and its subsequence converge to the same limit?We now answer this question with the following theorem.

Theorem 3.4.1 If a sequence $\{a_n\}$ converges to a limit,then the limit is unique.

Proof: Suppose,for contradiction,that the sequence $\{a_n\}$ converges to two limits a and b,$a \neq b$.Then given any $\varepsilon > 0$ there exists N(ε) $\in \mathbb{N}$ such that

$$|a_n - a| < \frac{\varepsilon}{2} \; \forall n \geqslant N(\varepsilon).$$

Also,there exists $\overline{N}(\varepsilon) \in \mathbb{N}$ such that

$$|a_n - b| < \frac{\varepsilon}{2} \; \forall n \geqslant \overline{N}(\varepsilon).$$

Hence,for all $n \geqslant$max$\{N(\varepsilon)$,$\overline{N}(\varepsilon)\}$,both inequalities $|a_n - a| < \frac{\varepsilon}{2}$ and $|a_n - b| < \frac{\varepsilon}{2}$ hold.Then

$$\begin{aligned} |a - b| &= |a - a_n + a_n - b| \\ &< |a - a_n| + |a_n - b| \\ &< \varepsilon \; for \; all \; n \geqslant max\{N(\varepsilon), \overline{N}(\varepsilon)\}. \end{aligned}$$

This implies that $a = b$.Contradiction.Hence the theorem is proved.

Having answered the question above using the above theorem,we crown it with the following theorem which makes the answer clearer.

Theorem 3.4.2 A sequence $(a_n) \in \Re$ is convergent to the limit a^* if and only if all of its subsequences converge to the same limit.

Proof:($\Longrightarrow$) Let (a_n) be a convergent sequence and suppose it converges to a^*.Let (a_{n_k}) be an arbitrary subsequence of (a_n).We prove (a_{n_k}) also converges to a^*.But from the definition of $a^*(=lim_{n\longrightarrow\infty} a_n)$ we have that $\forall \varepsilon > 0, \exists$ an integer $n(\varepsilon) > 0$ such that

$$|a_n - a^*| < \varepsilon \; \forall n \geqslant n(\varepsilon).$$

Then,for any $k > n(\varepsilon)$,and by induction hypothesis,$n_k \geqslant k > n(\varepsilon)$ so that,replacing n by n_k,we obtain that $\forall k \geqslant n(\varepsilon)$,

$$|a_{n_k} - a^*| < \varepsilon.$$

This implies,$a_{n_k} \longrightarrow a^*$ as $k \longrightarrow \infty$.

($\Longleftarrow$) Suppose every subsequence of (a_n) converges to a^*.We want to prove (a_n) converges to a^*.But (a_n) is a subsequence of itself.The result follows.The proof is complete. $\Box$

The theorem is very useful in showing that a sequence is not convergent.By the theorem,it suffices to produce two subsequences of the given sequence which converges to different limits.For example,if

$$\{a_n\} = (-1)^n \ \forall n \geqslant 1$$

then the subsequence $\{1, 1, 1, ...\}$ converges to 1 and the subsequence $\{-1, -1, -1, ...\}$ converges to -1.Hence,

$$(-1)^n \ \forall n \geqslant 1$$

is not convergent(by theorem 3.5.1).

If we cannot compute a limit directly(when it exists),we may be able to find it indirectly by means of a theorem called the Sandwich Theorem.

3.5 Sandwich Theorem

This theorem refers to a sequence $\{b_n\}$ whose nth term is sandwiched between the nth terms of two other convergent sequences $\{a_n\}$ and $\{c_n\}$.If $\{a_n\}$ and $\{c_n\}$ have the same limit as $n \longrightarrow \infty$,then $\{b_n\}$ has this limit also.We now prove a special case of the Sandwich theorem in which $\{a_n\}$ is the constant sequence $a_n = l \ \forall n$.

Theorem 3.5.1 Sandwich Theorem Suppose $\{a_n\}$ and $\{b_n\}$ are sequences of real numbers such that for some integer $N_0 \geqslant 1$,we have

$$l \leqslant b_n \leqslant a_n \ \forall n \geqslant N_0.$$

If $\{a_n\}$ converges to l,then $\{b_n\}$ also converges to l.

Proof: By definition,$a_n \longrightarrow l$ means given any $\varepsilon > 0$,there exists an integer $\overline{n} > 0$ such that $|a_n - l| < \varepsilon$ for all $n > \overline{n}$. Observe that $0 \leqslant b_n - l$,so that since $b_n \leqslant a_n$ we have $0 \leqslant |b_n - l| \leqslant |a_n - l|$. So,for given $\varepsilon > 0$, and all $n >$max$(\overline{n}, N_0)$,we have

$$|b_n - l| \leqslant |a_n - l| < \varepsilon.$$

Hence,$b_n \longrightarrow l$ as $n \longrightarrow \infty$. $\square$

Next,we look at the definition of a minimizing sequence in finite dimensional spaces.

3.6 Minimizing sequence

Let X be a compact subset of $\Re^n$ and $F : X \longrightarrow \Re$ a continuous map.

Since X is compact,we can define a sequence (x_n) on X that converges to a point of X.

(x_n) convergent implies limit exists,such that

$$lim_{n\longrightarrow\infty}F(x_n) = inf_{x\in K}F(x).$$

Hence,(x_n) is a minimizing sequence.

Proof:

K bounded,implies for arbitrary α,

$$inf_{x\in K}K = \alpha$$

then $\forall\varepsilon = \frac{1}{n} > 0, \exists\ (x_n) \subseteq K$ such that

$$\alpha \leqslant F(x_n) < \alpha + \frac{1}{n}.$$

Taking limit as $n \longrightarrow \infty$,

$$\alpha \leqslant lim_{n\longrightarrow\infty}F(x_n) < \alpha.$$

Then by Sandwich theorem,

$$lim_{n\longrightarrow\infty}F(x_n) = \alpha$$

$$\Rightarrow lim_{n\longrightarrow\infty}F(x_n) = inf_{x\in K}F(x)$$

□

Another important concept is that of a function being coercive which will be important in proceeding chapter.

3.7 Coercivity of a function

Let $F : \Re^n \longrightarrow \Re$ be a real valued function.F is said to be coercive if

$$lim_{\|x\|\longrightarrow\infty}F(x) = +\infty$$

Lastly,we are going to consider the difference between local and global minimizers on finite dimensional spaces.

3.8 Local and Global minimizers

We say that a point $\overline{u} \in V$ is a local minimizer of F in V if there exists a positive constant $r > 0$ such that

$$F(\overline{u}) \leqslant F(v)\ \forall v \in B(\overline{u}, r).$$

It is a global minimizer if the inequality holds for all points v in V.

Nevertheless,we are going to use the definition of global minimization in this setting.

Now,if $\overline{u}$ is a global minimum of F in V,we just write

$$F(\overline{u}) = min_{v\in K}F(v)$$

Chapter 4

Some Minimization Existence Theorems on Finite Dimensional Spaces

This chapter discusses the main existence theorems of minimizers in finite dimensional spaces.We however use the basic concepts and definitions given in previous chapters to prove our result in this discourse.

To define minimizers in this space,we consider the following existence theorems explicitly.

4.1 Existence Theorems

Theorem 4.1.1 Let X be a normed linear space,$K \subseteq X$,$K \neq \emptyset$ and $F : X \longrightarrow \Re \cup \{+\infty\}$.
If F is continuous and K is compact,then $\exists\ \overline{x} \in K$ such that

$$F(\overline{x}) \ = min_{x \in K} F(x).$$

Proof:
Let $(x_n) \subseteq K$ be a minimizing sequence.Then

$$lim_{n \longrightarrow \infty} F(x_n) = inf_{x \in K} F(x). \tag{4.1}$$

(from Definition 3.6)
K compact,implies closedness,we can define a subsequence that converges to a point in K.That is,

$$\exists\ n_k : x_{n_k} \longrightarrow \overline{x} \in K.$$

$$\Rightarrow F(x_{n_k}) \longrightarrow F(\overline{x}) \in K.$$

$$\Rightarrow\ lim_{k \longrightarrow \infty} F(x_{n_k}) = F(\overline{x}). \tag{4.2}$$

From (1),using Bolzano Weierstrass theorem,$\exists$ a subsequence $(x_{n_k}) \subseteq (x_n)$ that converges to the same limit,hence,

$$lim_{k \longrightarrow \infty} F(x_{n_k}) = inf_{x \in K} F(x). \tag{4.3}$$

Hence by uniqueness of limit,

(2) and (3) $\Rightarrow$

$$F(\overline{x}) = inf_{x \in K} F(x).$$

Since $\overline{x} \in K$,

$$F(\overline{x}) = min_{x \in K} F(x).$$

Hence there exists a minimizing sequence x_n,where $\overline{x}$ is a global minimum of F in K.

To consider the next theorem concerning the existence of minimizers in this space,we consider the following Lemma which we shall give here without proof.

Lemma 4.1.2 Let $K \neq \emptyset$,closed subset of X.If F is coercive and continuous on some open set containing K,then the followings hold

(i) The function F is bounded from below on K.

(ii) Any minimizing sequence of F in K is bounded.

With this,we proceed to the next theorem.

Theorem 4.1.3 Let X be a normed linear space such that $dim(X) < +\infty$. Let K $\subseteq X$, $K \neq \emptyset$, closed and $F : K \longrightarrow \Re$ continuous .

If K is bounded or F is coercive (i.e.,$lim_{\|x\| \longrightarrow \infty} F(x)$ =$+\infty$) ,then $\exists\ \overline{x} \in K$:

$$F(\overline{x}) = min_{x \in K} F(x).$$

Proof:

Let (x_n) be a minimizing sequence,

then,

$$lim_{n \longrightarrow \infty} F(x_n) = inf_{x \in K} F(x) \tag{4.4}$$

Claim:(x_n) is bounded.

Proof of claim:

Assume for contradiction that (x_n) is unbounded .This implies,(x_n) is neither bounded above nor below,hence,(x_n) not bounded above implies

$$\exists\ n_k : lim_{k \longrightarrow \infty} \| x_{n_k} \| = +\infty.$$

But F being coercive implies that

$$lim_{k\longrightarrow\infty}F(x_{n_k}) = \dagger\infty \tag{4.5}$$

From (4),using Bolzano Weierstrass theorem,$\exists$ a subsequence $(x_{n_k}) \subseteq (x_n)$ that converges to the same limit,hence,

$$lim_{k\longrightarrow\infty}F(x_{n_k}) = inf_{x\in k}F(x) \tag{4.6}$$

Hence,by uniqueness of limit,(5) and (6)implies

$$inf_{x\in K}F(x) = +\infty.$$

$$\Rightarrow F(x) = +\infty \ \forall x \in K.$$

This is a contradiction because of the fact that F is bounded from below on K(from Lemma 4.1.2)

Hence (x_n) is bounded.

Using Bolzano Weierstrass theorem,

$$\exists \ x_{n_k} : x_{n_k} \longrightarrow \overline{x}$$

$$\Rightarrow F(x_{n_k}) \longrightarrow \ F(\overline{x}) \tag{4.7}$$

By uniqueness of limit,(6) and (7) implies

$$F(\overline{x}) = inf_{x\in K}F(x).$$

Since $\overline{x} \in k$,

$$F(\overline{x}) = min_{x\in K}F(x).$$

Hence there exists a minimizing sequence x_n, where $\overline{x}$ is a global minimum of F in K.

Now,we are going to deal with the special case in $\Re^n$.We however prove the following corollary as an easy exercise.

Corollary 4.1.4 Let K $\subseteq \Re^n$, $F : \Re^n \longrightarrow \Re$, continuous and K is closed and non empty.If K is bounded or F is coercive,then $\exists \ \overline{x} \in K$ such that

$$F(\overline{x}) = min_{x\in K}F(x).$$

Proof:

Let (x_n) be a minimizing sequence of F in K.By proposition,$(x_n) \subseteq$ K is bounded.So by Bolzano-Weierstrass theorem,(x_n) has a subsequence (x_{n_k}) that converges to some point $\overline{x} \in \Re^n$. Since K is closed,then $\overline{x} \in K$. Using the continuity of F at $\overline{x}$,it follows that

$$lim_{k\longrightarrow\infty}F(x_{n_k}) = F(\overline{x}). \tag{4.8}$$

On the other hand,since $F(x_{n_k})$ is a subsequence of $F(x_n)$,we have

$$lim_{k\longrightarrow\infty}F(x_{n_k}) = inf_{x\in K}F(x) \tag{4.9}$$

Using (8) and (9) and the uniqueness of limit,it follows that $F(\overline{x}) = inf_{x\in K}F(x)$.

$Since\ \overline{x} \in K, we\ have\ that$

$$F(\overline{x}) = min_{x\in K}F(x).$$

So, $\overline{x}$ is a global minimum of F in K.

4.2 Conclusion

We note however that conditions used for the existence of minimizers in finite dimensional spaces are not applicable to infinite dimensional spaces.This is because of the fact that in infinite dimensional spaces,closed and bounded subsets need not be compact.

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