Matrix Transformations Into The Generalized Space of Entire Sequences

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Abstract

The object of this note is to characterize infinite matrices between some sequence spaces and the generalized set of entire sequences. The investigations reveal that the sets Γ and $c_0(1/k)$ are essentially the same. Their generalized classes, $(c_0^v(p,s),:\Gamma(\mathrm{p}))$ and $(l^v(p,s):\Gamma(\mathrm{p}))$ are characterized.

Key Words: Duals, Entire Sequences, Matrix Transformations, Paranormed Spaces, Sequence Spaces

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1. Introduction

1.1 Matrix transformations

Let $A = (a_{nk})$ be an infinite matrix of complex numbers a_{nk} $(n, k = 1, 2, \ldots)$ and X, Y be two nonempty subset of the space ω of all complex sequences. The matrix A is said to define a matrix transformation from X into Y and write $A : X \to Y$ if for every $x = (x_k) \in X$ and every integer n we have

$$A_n(x) = \sum_{k=1}^{\infty} a_{nk} x_k.$$

If the sequence $Ax = (A_n(x))$ exists, then it is called the transformation of x by the matrix A. Further, $A \in (X, Y)$ if and only if $A_n \in X^{\beta}$ for all $Ax \in Y$, whenever $x \in X$; where the pair (X, Y) denotes the class of matrices A. The determination of the necessary and sufficient conditions for a matrix $A = (a_{nk})$ to be in the class (X, Y) for varying sequence spaces X and Y has been the focal point of many researchers.

1.2 Some new sequence spaces: Definitions and notations

Take $p = (p_k)$, $p_k > 0$ for all k and let $q = (q_k)$ be any bounded sequence. Define any fixed sequence of non – zero complex numbers $v = (v_k)$ such that

$$\lim_{k\to\infty} \inf |\, v_k \,|^{1/k} = \eta, \ (0 < \eta < \infty).$$

The following sequence spaces are relevant in this work:

(a) $\Gamma(\mathrm{p}) = \{x = (x_k) : |\, k!\ x_k \,|^{q_k} \to 0,$ as $k \to \infty$. This is a linear metric space under the metric topology generated by the paranorm, $(f) = sup_k \,|\, k!\ x_k \,|^{q_k/M}$, (see [2]).

(b) $l^{v}(p, s) = \{x = (x_k) : sup_k k^{-1} \,|\, x_k v_k \,|^{p_k} < \infty, \ s \geq 0\}$. This space is paranormed by

$$h(x) = \left(\sum_k k^{-s} |\, x_k v_k \,|^{p_k}\right)^{1/M}.$$

(c) $c_0^{v}(p, s) = \{x = (x_{k)} : k^{-1} |\, x_k v_k \,|^{p_k} \to 0, \ s \geq 0\}$, paranormed by

$$g(x) = \sup_k (k^{-1} \,|\, x_k v_k \,|^{p_k})^{1/M}$$

where,

$$H = \sup_k p_k \text{ and } M = \max(1, H), \text{ see [1].}$$

If E is a set of complex sequences $x = (x_k)$ then E^{+} will denote the generalized Kőthe-Toeplitz dual of E defined by

$$E^{+} = \{a = (a_k) \in \omega : \sum_{k=1}^{\infty} a_k x_k \ converges \ \forall \, x \in E\}$$

If E is a set of complex sequences $x = (x_k)$ then E^{α} will denote the α- dual of E defined by

$$E^{\alpha} = \{a = (a_k) \in \omega : \sum_{k=1}^{\infty} |\, a_k x_k \,| < \infty, \ \forall \, x \in E \text{ (see [3])}$$

Further, if $E \subset \omega$, and E is a Kőthe space, then E is solid; and if E is solid then $E^{\alpha} = E^{\beta} = E^{\gamma}$ called the $\alpha-$, $\beta -$ and $\gamma -$ duals of E, respectively. That E is solid or total means when $x \in E$ and $|\, y_k \,| \leq |x_k\,|$, $\forall k \in N$ together imply $y \in E$, (see [4] and [5]).

Let $X \supset \emptyset$ be a BK- space. Then there is a linear one-to-one mapping $T : X^{\beta} \to X^{*}$; we denote this by saying $X^{\beta} \supset X^{*}$. $\emptyset$ is a set of finite sequences and X^{*} the continuous dual of X; while a BK- space is a vector space whose elements are complex sequences $x = (x_k)_{k\geq 0}$ and which is also a Banach space (that is, normed and complete) with continuous coordinates (that is, $||\, x^n - x \,||_X \to 0$ implies $|\, x^n - x \,| \to 0$ for each k, as $n \to \infty$), (see [6] and [7])

2. Some known results

The following known results play vital role in our main results, they amount to computing $\alpha -$ and continuous duals of the sequence spaces $l^v(p,s)$ and $c_0^v(p,s)$.

Lemma 1 (Lemma 2.1, [2]): Let $0 < p_k \leq sup_k\, p_k < \infty$. Then

(i) $(c_0^v(p,s))^\alpha = M_0^v(p,s)$,

where,

$$M_0^v(p,s) = \cup_{N>1}\{\, a = (a_k) \in \omega : \sum_k |\, a_k v_k^{-1}\, | k^{s/p_k} N^{-1/p_k} < \infty, s \geq 0\}$$

(ii) $(c_0^v(p,s))^*$ is isomorphic to $M_0^v(p,s)$

Lemma 2 (Lemma 2.2 [2]): (i) If $0 < p_k \leq sup_k < \infty$ and $p_k^{-1} + q_k^{-1} = 1$, $k = 1, 2,$ ... Then

(i) $(l^v(p,s))^\alpha = M^v(p,s)$,

(ii) $(l^v(p,s))^*$ is isomorphic to $M^v(p,s)$,

where,

$$M^v(p,s) = \{a = (a_k) \in \omega : \sum_k \,|\, a_k v_k^{-1}\, |^{q_k} k^{s(q_k-1)} N^{-q_k/p_k} < \infty, s \geq 0\}$$

3. Main Results

In what follows we prove the following theorems:

Theorem A: Let $0 < p_k \leq sup_k < \infty$ and $p_k^{-1} + q_k^{-1} = 1$, $k = 1, 2,$ … . Then $A \in (c_o^v(p,s) : \Gamma(p))$ if and only if

$$(n! \sum_k \,|\, a_k v_k^{-1}\, |\, M^{-/p_k} k^{s/p_k})^{q_n} \to 0, \text{ as } n \to \infty, M > 1,\; M \in N \tag{1}$$

Proof: For sufficiency, since $x \in c_o^v(p,s)$, there exists $M > 1$ such that

$$|\, v_k x_k\, | < M^{-/p_k} k^{s/p_k}, \forall\, k.$$

Let (1) hold, then for a given $\varepsilon > 0$, there exists an integer n_0 such that

$$(n! \sum_k \,|\, a_k v_k^{-1}\, |\, M^{-/p_k} k^{s/p_k})^{q_n} < \varepsilon, \forall\; n > n_0 \tag{2}$$

Now,

$$\begin{aligned}(n!\, A_n(x))^{q_n} &\leq (n! \textstyle\sum_{k=1}^{\infty} a_{nk} x_k)^{q_n}\\ &\leq (n! \textstyle\sum_{k=1}^{\infty} (a_{nk} v_k^{-1}) v_k^{-1} x_k)^{q_n}\\ &\leq (n! \textstyle\sum_{k=1}^{\infty} |a_{nk} v_k^{-1}|\, k^{s/p_k} M^{-1/p_k})^{q_n}\\ &\to 0 \text{ as } n \to \infty \text{ for } n \geq n_0 \text{ (by (1)}\end{aligned}$$

Necessity: If (1) does not hold, then there exist subsequences of (n) such that

$$(n! \sum_{k=1}^{\infty} |a_{nk} v_k^{-1}|\, k^{s/p_k} M^{-1/p_k})^{q_n} > \varepsilon \text{ when } n \to \infty \tag{3}$$

Since $A \in (c_o^v(p,s) : \Gamma(p))$, then the sequence $A_n = (a_{nk})_{k=0}^{\infty} \in (c_o^v(p,s))^*$. So by Lemma (1)

$$\sum_{k=1}^{\infty} |a_{nk} v_k^{-1}|\, k^{s/p_k} M^{-1/p_k}\, \infty, \text{ for } M > 1 \tag{4}$$

Since $x = e^k \in (c_o^v(p,s),\; A_n = (a_{nk}) \in \Gamma(p)$, so that,

$$(n!\, |a_{nk} v_k^{-1}|)^{q_n} \leq A_k\; \forall\, n \text{ and for each fixed } k \tag{5}$$

Let us construct a sequence $(x_k) \in (c_o^v(p,s)$ and show that the corresponding sequence $(A_n) \notin \Gamma(p)$. This will amount to provision that the condition is necessary.

By (3) $n = n_1$ and $k = q_1$ can be chosen such that

$$(n_1! \sum_{k=1}^{q_1} |a_{n_1 k} v_k^{-1}|\, (M+1)^{-1/p_k} k^{s/p_k})^{q_{n_1}} > 1 \tag{6}$$

After fixing n_1 by (4) we choose $k = k_1 > q_1$ such that

$$(n_1! \sum_{k=k_1+1}^{\infty} |a_{n_1 k} v_k^{-1}|\, (M+1)^{-1/p_k} k^{s/p_k})^{q_{n_1}} < \varepsilon \tag{7}$$

Taking for all n, defined by

$$x_k = \begin{cases} \mathrm{sgn}|a_{nk} v_k^{-1})(M+1)^{-1/p_k}\ v_k\ k^{s/p_k} \text{ for all } n, \text{and } 1 \le \mathrm{k} \le \mathrm{k}_1 \\ \mathrm{sgn}|a_{nk} v_k^{-1})(M+1)^{-1/p_k}\ v_k\ k^{s/p_k} \text{ for all } n, \text{and } \mathrm{k}_{j-1} \le \mathrm{k} \le \mathrm{k}_j,\ \ j = 2, 3, \dots \end{cases}$$

(8)

so that $(x_k) \in (c_o^v(p,s)$ and

$$(M+i)^{-1/p_k} \le (M+i-1)^{-1/p_k} \qquad (9)$$

Thus, using (6), (9) and (7), we should have

$$(n_1)!\,|\,A_{n_1}\,|)^{q_{n_1}} \ge (n_1!\,|\sum_{k=1}^{k_1}(a_{n_1k}v_k^{-1})v_k x_k|)^{q_{n_1}} - (n_1!\,|\sum_{k=k_1+1}^{\infty}(a_{n_1k}v_k^{-1})v_k x_k|)^{q_{n_1}}$$
$$\ge (n_1!\,|\sum_{k=1}^{k_1}(a_{n_1k}v_k^{-1})(M+1)^{-1/p_k}\ k^{s/p_k}\ |)^{q_{n_1}} -$$
$$(n_1!\,|\sum_{k=k_1+1}^{\infty}(a_{n_1k}v_k^{-1})\ (M+2)^{-1/p_k}\ \ k^{s/p_k}\ |)^{q_{n_1}}$$

➢ $1-\varepsilon$.

Thus, from (5) and (9), we must have for all n,

$$(n_1!\,|\sum_{k=1}^{k_i}(a_{n_1k}v_k^{-1})(M+i)^{-1/p_k}\ k^{s/p_k}\ |)^{q_{n_i}} \le$$
$$(n_1!\,|\sum_{k=1}^{k_1}(a_{n_1k}v_k^{-1})\ (M)^{-1/p_k}\ \ k^{s/p_k}\ |)^{q_{n_i}}$$
$$\le c_{k_i};$$

where,

$$c_{k_i} = \sum_{k=1}^{k_i} A_k \qquad (10)$$

By (3) $n = n_2 > n_1$ and $q_2 > k_1$ can be chosen such that

$$(n_1!\,|\sum_{k=1}^{q_2}(a_{n_1k}v_k^{-1})(M+2)^{-1/p_k}\ k^{s/p_k}\ |)^{q_{n_2}} > 2 + \le c_{k_1} \qquad (11)$$

Having fixed n_2, by (4) choose $k = k_2 > q_1$ such that

$$(n_1!\,|\sum_{k=k_2+1}^{\infty}(a_{n_1k}v_k^{-1})(n_2!\,|\sum_{k=k_1+1}^{k_2}(a_{n_2k}v_k^{-1})v_k x_k|)^{q_{n_2}}|)^{q_{n_2}} < \varepsilon \qquad (12)$$

$$(n_2)!\,|\,A_{n_2}\,|)^{q_{n_2}} \le (n_2!\,|\sum_{k=k_1+1}^{k_2}(a_{n_2k}v_k^{-1})v_k x_k|)^{q_{n_2}} - (n_2!\,|\sum_{k=1}^{k_1}(a_{n_2k}v_k^{-1})v_k x_k|)^{q_{n_2}}$$
$$-(n_2!\,|\sum_{k=k_2+1}^{\infty} a_{n_2k} v_k x_k|)^{q_{n_2}}$$
$$\ge (n_2!\,|\sum_{k=k_1+1}^{k_2}(a_{n_2k}v_k^{-1})(M+2)^{-1/p_k}\ k^{s/p_k})^{q_{n_2}}$$
$$-(n_2!\,|\sum_{k=1}^{k_1}(a_{n_2k}v_k^{-1})(M+1)^{-1/p_k}\ k^{s/p_k})^{q_{n_2}}$$
$$-(n_2!\,|\sum_{k=k_2+1}^{\infty} a_{n_2k}(M+3)^{-1/p_k}\ k^{s/p_k}|)^{q_{n_2}} \qquad \text{[by (8)]}$$

$> 2 - \varepsilon$ [by (9), (10), (11), (12)].

Continuously proceeding in this manner, we can choose $n_i > n_{i-1}$ and $q_i > k_{i-1}$ by (3) such that

$$(n_i!\,|\sum_{k=k_{i-2}+1}^{k_i}(a_{n_ik}v_k^{-1})(M+i)^{-1/p_k}\ k^{s/p_k})^{q_{n_i}} > i + c_{k_{i-1}}.$$

Therefore, for fixed n_i, we can choose $k_i > q_i$ by (4) such that

$$(n_i!\,|\sum_{k=k_i+1}^{\infty}(a_{n_ik}v_k^{-1})(M+i)^{-1/p_k})^{q_{n_i}} < \varepsilon$$

So, as above by the use of (8), (9) and (10) it can shown that

$$(n_i!\,|\,A_{n_i}\,|)^{q_{n_i}} > i - \varepsilon.$$

But ε was arbitrarily given so that $(n_i!\,|\,A_{n_i}\,|)^{q_{n_i}} \to \infty$ as $n \to \infty$. Hence the sequence $(A_n) \notin \Gamma(p)$. This proves that (1) is a necessity.

Theorem B: Let $0 < \mathrm{p_k} \le \sup_\mathrm{k} < \infty$ and $\mathrm{p_k^{-1}} + \mathrm{q_k^{-1}} = 1$, $\mathrm{k} = 1, 2, \dots$. Then $\mathrm{A} \in (\,l^v(p,s) : \Gamma(p))$ if and only if

$$(n!\sum_k\ |\,a_{nk}v_k^{-1}\,|^{q_k}\ k^{s(q_k-1)}N^{-q_k/p_k})^{q_n} \to 0, \text{ as n} \to \infty, \text{ uniformly in k}, \qquad (13)$$

where,

$$p_{k>1} \text{ and } p_k^{-1} + q_k^{-1} = 1.$$

Proof: Sufficiency– Since $(x_k) \in l^v(\mathrm{p}, \mathrm{s})$, then there exists a finite $M \geq 1$ such that

$$\sum_k k^{-s} |x_k v_k|^{p_k} \leq M \quad (14)$$

Let (13) hold good. Then given an $\varepsilon > 0$, there exists some integer $N = N(\varepsilon)$ independent of k such that

$$(n! \sum_k |a_{nk} v_k^{-1}|^{q_k} k^{s(q_k-1)} N^{-q_k/p_k})^{q_n} < \frac{\varepsilon}{M}, \ \forall n \geq N \quad (15)$$

Now,

$$(n! A_n(x))^{q_n} \leq (n! \sum_{k=1}^{\infty} |a_{nk} x_k|^{q_n}$$
$$\leq (n! \sum_{k=1}^{\infty} |a_{nk} v_k^{-1}| |v_k^{-1} x_k|)^{q_n}$$
$$\leq (n! \sum_k |a_{nk} v_k^{-1}| |v_k^{-1} x_k| k^{s/p_k} k^{-s/p_k} N^{-q_k/p_k})^{q_n}$$
$$\leq (n! \sum_k |a_{nk} v_k^{-1}|^{q_k} |v_k^{-1} x_k| k^{s/p_k} k^{-s/p_k} N^{-q_k/p_k})^{q_n}$$
$$\leq (n! \sum_k |a_{nk} v_k^{-1}|^{q_k} |v_k^{-1} x_k| k^{s(q_k-1)} N^{-q_k/p_k})^{q_n}$$
$$\cdot (\sum_k |v_k x_k|^{p_k} k^{-s})^{q_n/p_k}$$

$$\leq (\varepsilon/M)^{1/q_k} \cdot M^{q_n/p_k}$$
$$< \varepsilon.$$

Since the choice of ε was arbitrary, it shows that $A \in \Gamma(p)$.

Necessity– If (13) does not hold, ten there exist subsequences of values of n such that

$$(n! \sum_k |a_{nk} v_k^{-1}|^{q_k} k^{s(q_k-1)} N^{-q_k/p_k})^{q_n} \geq \varepsilon$$

(16)

Since the matrix between, $l^v(p,s)$ and $\Gamma(p)$ being $BK-$ spaces, is continuous, the sequence $(a_{nk}) \in (l^v(p,s))^*$. Hence, by Lemma 2,

$\sum_k |a_{nk} v_k^{-1}|^{q_k} k^{s(q_k-1)} N^{-q_k/p_k}$ is convergent for $N > 1$

(17)

When $x_k = 1$ and $x_j = 0$ for $j \neq k$, $x_k \in l^v(p,s)$ so that $A_n = (a_{nk})_{k=1}^{\infty} \in \Gamma(p)$. Hence,

$(n! |a_{nk} v_k^{-1}|)^{q_n} \leq A_k'$, for all n and each fixed k

(18)

This implies that

$(n! |a_{nk} v_k^{-1}| k^{s/p_k})^{q_n} \leq A_k$, where $A_k = k^{s/p_k} A_k'$, for each fixed k and for all n.

Using (16), (17) and (18), we can construct a sequence $(x_k) \in l^v(p,s)$ and show that $(A_n(x)) \notin \Gamma(p)$, then that will suffice to show the necessity of condition holds.

Now, by (16) choose $n = n_1$ and $k = q_1$ such that

$$(n_1! \sum_{k=1}^{q_1} |a_{n_1 k} v_k^{-1}|^{q_k} k^{s(q_k-1)} N^{-q_k/p_k})^{q_{n_1}} > 1 \quad (19)$$

Having fixed n_1, by (17), for $\varepsilon > 0$, we can choose $k_1 > q_1$ such that

$$(n_1! \sum_{k=k_1+1}^{\infty} |a_{n_1 k} v_k^{-1}|^{q_k} k^{s(q_k-1)} N^{-q_k/p_k})^{q_{n_1}} < \varepsilon$$

(20)

the series being convergent.

Let $x_k = |a_{n_1 k} v_k^{-1}|^{q_k-1} k^{s(q_k-1)} N^{-q_k/p_k} \operatorname{sgn}(a_{n_1} v_k^{-1})$, for $1 \leq k \leq k_1$, then

$$|n_1! A_{n_1}(x)|^{q_{n_1}} \geq (|n_1! \sum_{k=1}^{k_1} (a_{n_1 k} v_k^{-1}) x_k|)^{q_{n_1}} - (n_1! |\sum_{k=k_1+1}^{\infty} (a_{n_1 k} v_k^{-1}) x_k|)^{q_{n_1}}$$
$$\geq (|n_1! \sum_{k=1}^{k_1} (a_{n_1 k} v_k^{-1}) x_k k^{s(q_k-1)} N^{-q_k/p_k}|)^{q_{n_1}}$$
$$-(n_1! |\sum_{k=k_1+1}^{\infty} (a_{n_1 k} v_k^{-1}) x_k k^{s(q_k-1)} N^{-q_k/p_k}|)^{q_{n_1}} \cdot$$
$$(\sum_{k=k_1+1}^{\infty} |x_k|^{p_k} k^{-s})^{q_{n_1}/p_k}$$
$$> 1 - \varepsilon$$

Since, $(q_k - 1) = q_k/p_k$, from (17) we have for all n,

$$(n_1! \sum_{k=1}^{k_1} |a_{n_1 k} v_k^{-1}|^{q_k} k^{s(q_k-1)} N^{-q_k/p_k})^{q_n} \leq (n_1! \sum_{k=1}^{k_1} |a_{n_1 k} v_k^{-1}| k^{s/p_k} N^{-q_k/p_k})^{q_k/q_n}$$

$$\leq A_1^{q_k} + A_2^{q_k} + \dots A_{k_1}^{q_k}$$

$$\leq c_{k_1}, \text{ where } c_{k_1} = A_1^{q_k} + A_2^{q_k} + \dots A_{k_1}^{q_k} \quad (22)$$

Now by (15), choose $n_2 > n_1$ and $q_2 > k_1$ such that

$$(n_1! \sum_{k=k_1+1}^{q_2} | a_{n_2 k} v_k^{-1} |^{q_k} k^{s(q_k-1)} N^{-q_k/p_k})^{q_{n_2}} > 2 + c_{k_1}$$

(23)

Having fixed n_2, by (16), it is possible to choose a $k_2 > q_2$ such that

$$(n_2! \sum_{k=k_1+1}^{\infty} | a_{n_2 k} v_k^{-1} |^{q_k} k^{s(q_k-1)} N^{-q_k/p_k})^{q_{n_2}} < \varepsilon$$

(24)

Again, let $x_k = | a_{n_2 k} v_k^{-1} |^{q_k - 1} k^{s(q_k-1)} N^{-q_k/p_k} \mathrm{sgn}(a_{n_2 k} v_k^{-1})$, for $1 \leq k \leq k_2$, then we have

$$|n_2! A_{n_2}(x)|^{q_{n_2}} \geq (|n_2! \sum_{k=k_1+1}^{k_2} (a_{n_2 k} v_k^{-1}) x_k |)^{q_{n_2}} - (n_2! |\sum_{k=1}^{k_1} (a_{n_2 k} v_k^{-1}) x_k |)^{q_{n_2}}$$

$$- (n_2! |\sum_{k=k_1+1}^{\infty} (a_{n_2 k} v_k^{-1}) x_k |)^{q_{n_2}}$$

$$\geq (n_2! \sum_{k=k_1+1}^{k_2} |a_{n_2 k} v_k^{-1} | k^{s(q_k-1)} N^{-q_k/p_k})^{q_{n_2}}$$

$$- (n_2! \sum_{k=1}^{k_1} |a_{n_2 k} v_k^{-1} | |x_k |)^{q_{n_2}}$$

$$- (n_2! \sum_{k=k_2+1}^{\infty} |a_{n_2 k} v_k^{-1} | |x_k |)^{q_{n_2}}$$

$$> 2 + c_{k_1} - c_{k_1} - (n_2! \sum_{k=k_2+1}^{\infty} |a_{n_2 k} v_k^{-1} |^{q_k} k^{s(q_k-1)} N^{-q_k/p_k})^{q_{n_2}/q_k} .$$

$$(\sum_{k=k_2+1}^{\infty} | x_k |^{p_k} k^{-s})^{q_{n_2}/p_k}$$

$> 2 - \varepsilon$, by (22), (23) and (24)

Proceeding in this manner, by (16), we can choose $n_m > n_{m-1}$ and $q_m > k_{m-1}$ such that

$$(n_m! \sum_{k=k_{m-1}+1}^{q_m} |a_{n_m k} v_k^{-1} | k^{s(q_k-1)} N^{-q_k/p_k})^{q_{n_m}}$$

$$> m + (m-1) c_{k_1} + (m-2) c_{k_2} + \cdots + c_{k_{m-1}}$$

(25)

Having fixed n_m by (17), choose $k_m > q_{m-1}$ such that

$$(n_m! \sum_{k=k_m+1}^{\infty} |a_{n_m k} v_k^{-1} |^{q_k} k^{s(q_k-1)} N^{-q_k/p_k})^{q_{n_m}} < \varepsilon \quad (26)$$

Finally, take $x_k = | a_{n_m k} v_k^{-1} |^{q_k - 1} k^{s(q_k-1)} N^{-q_k/p_k} \mathrm{sgn}(a_{n_m k} v_k^{-1})$, for $k_{m-1} \leq k \leq k_m$, then we should have,

$$|n_m! A_n(x)|^{q_{n_m}} \to \infty \text{ as } n \to \infty$$

Hence, $(A_n(x)) \notin \Gamma(x)$, so that (13) is necessary.

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